Necessary Bandwidth for High-Index FM

ADEL A. ALI AND MOHAMMED A. ALHAIDER

Abstract—Necessary bandwidth for high-index FM with Gaussian modulation is studied. On the basis of the quasi-static argument, a simple formula is devised for calculating the transmission bandwidth. It is shown that the proposed formula is more meaningful than Carson's rule, since it relates bandwidth to the transmission quality and the peak factor of the modulating signal.

Introduction

For frequency-modulated systems having large modulation indices, we shall use the quasi-static approach to approximate the power spectrum of the modulated signal. The method asserts that the probability density function of the modulating signal is a good approximant of the FM spectrum. Several proofs were presented in the literature [1]-[4], together with the discussion of its possible application. It has been shown that the method yields good results for slow, strong modulation [5], or, equivalently, when the modulation index m satisfies the condition

$$m = \frac{\text{rms frequency deviation}}{\text{bandwidth of modulation}} \gg 1. \tag{1}$$

In general, communication satellite systems have modulation parameters that meet those required for the quasi-static argument [6].

To calculate the bandwidth necessary for transmitting the FM signal, Carson's rule is used as a rule of thumb, yielding acceptable results for most FM systems. The rule suggests that the necessary bandwidth B_n is twice the sum of the peak frequency deviation ΔF and the highest modulating frequency f_m, namely,

$$B_n = 2(f_m + \Delta F). \tag{2}$$

Carson's rule does not relate the necessary bandwidth to the transmission fidelity expressed as the signal-to-band-limiting distortion ratio. On the other hand, there is no "peak frequency" for Gaussian modulation.

The purpose of this letter is to derive a simple expression relating the necessary bandwidth to the system's performance and the peak factor of the modulating signal. The peak factor given by the ratio of peak-to-rms frequency deviation does not exist for Gaussian modulation [7]. The peak factor is therefore redefined as the ratio of the value exceeded by the modulating signal, for a certain percentage of the time to the rms signal value.

Analysis

When the modulating signal has Gaussian statistics, as in the case of frequency-division multiplexed (FDM) voice channels, video signals, etc., the power spectral density $W(f)$ of the modulated signal is approximated by [8]

$$W(f) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-f^2/2\sigma^2). \tag{3}$$

Manuscript received April 21, 1982; revised September 27, 1982.

The authors are with the College of Engineering, King Saud University, Riyadh, Saudi Arabia.

0018-9219/83/0100-0184$01.00

TABLE I
PROBABILITY OF EXCEEDING A GIVEN PEAK FOR DIFFERENT PEAK VALUES α

Peak factor α (Equation (18))	$\sqrt{2}$	2.58	3.3	3.89	4.42	4.895	7
α (dB)	3	8.23	10.3	11.8	12.9	13.8	16.9
Probability that $\Delta F = \alpha\sigma$ is exceeded (Equation (20))	45%	1%	0.1%	0.01%	0.001%	0.0001%	2.5×10^{-10}%

The frequency is measured from that of the carrier—assumed to have unit power—and σ is the rms frequency deviation of the modulated carrier. If an ideal bandpass filter of bandwidth $2F$ is used to limit the spectrum, the resulting band-limiting distortion $D(F)$ can be defined as

$$D(F) \triangleq \frac{\text{total signal power} - \text{transmitted signal power}}{\text{total signal power}}. \tag{4}$$

The total power under $W(f)$ of (3) is normalized to unity; hence, (4) can be rewritten as

$$D(F) = 1 - \int_{-F}^{F} W(f)\, df \tag{5}$$

$$= 1 - \int_{-F}^{F} \frac{1}{\sigma\sqrt{2\pi}} \exp(-f^2/2\sigma^2)\, df \tag{6}$$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_{0}^{F/\sigma\sqrt{2}} \exp(-f^2/2\sigma^2)\, d(f/\sigma\sqrt{2})$$

$$= 1 - \operatorname{erf}(F/\sigma\sqrt{2}) \tag{7}$$

$$= \operatorname{erfc}(F/\sigma\sqrt{2}). \tag{8}$$

Assuming the band-limiting distortion is the only source of distortion, we see that the signal-to-distortion ratio is

$$\frac{S}{D} = \frac{1}{D(F)} = \frac{\text{total signal power fed to filter}}{\text{power not passed by the filter}} \tag{9}$$

which can be expressed in decibels as

$$\frac{S}{D}(\text{dB}) = -10 \log D(F) \tag{10}$$

and substituting from (8) into (10), we get

$$\frac{S}{D}(\text{dB}) = -10 \log \{\operatorname{erfc}(F/\sigma\sqrt{2})\}. \tag{11}$$

For large values of x, a lower bound on erfc x is given by [9]

$$\operatorname{erfc} x > \frac{1}{x\sqrt{\pi}} \exp(-x^2) \cdot \left\{1 - \frac{1}{2x^2}\right\} \tag{12}$$

and similar upper bound is [9]

$$\operatorname{erfc} x < \frac{1}{x\sqrt{\pi}} \exp(-x^2). \tag{13}$$

Substituting from (12) into (11) and rearranging, we obtain a lower bound on S/D

$$\frac{S}{D}(\text{dB}) > \left(0.98 + 10 \log \frac{F}{\sigma} + 2.174 \left(\frac{F}{\sigma}\right)^2\right) - 10 \log \left(1 - \left(\frac{\sigma}{F}\right)^2\right). \tag{14a}$$

Similarly, an upper bound on S/D is

$$\frac{S}{D}(\text{dB}) < 0.98 + 10 \log \frac{F}{\sigma} + 2.174 \left(\frac{F}{\sigma}\right)^2. \tag{14b}$$

An intermediate value of S/D can be found by interpolating halfway between the bounds of (14) to get

$$\frac{S}{D} = \left\{0.98 + 10 \log \frac{F}{\sigma} + 2.174 \left(\frac{F}{\sigma}\right)^2\right\} - 5 \log \left\{1 - \left(\frac{\sigma}{F}\right)^2\right\} \text{dB}. \tag{15}$$

The error in S/D as calculated from (15) is less than 1.5 dB for $F/\sigma = \sqrt{2}$ and decreases to less than 0.25 dB for $F/\sigma = 3$.

We now simplify (15) further and obtain a relation similar to Carson's rule by first rewriting (15) as

$$\frac{S}{D} = 0.98 + 5 \log \frac{(F/\sigma)^2}{1 - (\sigma/F)^2} + 2.174 (F/\sigma)^2 \text{ dB} \tag{16}$$

and then expanding the argument of the logarithm by the binomial theorem to get

$$\frac{S}{D} = 0.98 + 5 \log \{(F/\sigma)^2(1 + (\sigma/F)^2 + (\sigma/F)^4 + \cdots)\} + 2.174 (F/\sigma)^2 \text{ dB}$$

$$\frac{S}{D} \simeq 0.98 + 5 \log (1 + (F/\sigma)^2) + 2.174 (F/\sigma)^2 \text{ dB}, \quad \text{for } F \gg \alpha. \tag{17}$$

The foregoing equation relates S/D to the transmission bandwidth $2F$ for Gaussian modulation with rms frequency deviation α.

The peak factor α is defined as

$$\alpha = \frac{\Delta F}{\sigma} \tag{18}$$

with ΔF being the value exceeded by the modulating signal for a prescribed fraction of the time. For Gaussian modulation, the probability $p(\Delta F)$ that the value ΔF is exceeded is

$$p(\Delta F) = \frac{2}{\sigma\sqrt{2\pi}} \int_{\Delta F}^{\infty} \exp(-t^2/2\sigma^2)\, dt \tag{19}$$

$$= \frac{2}{\sqrt{\pi}} \int_{\Delta F/\sigma}^{\infty} \exp(-v^2/2)\, dv \tag{20}$$

where $v = t/\sigma$.

The probability integral of (20) is tabulated in [10]. A common value for the time percentage over which the peak is exceeded is 0.01 percent, yielding $\Delta F/\sigma = 3.89$ (11.8 dB). Inclusion of 0.001 percent peaks increases the peak factor by only 1.1 dB [7]. Table I depicts a range of values of the peak factor, corresponding to peaks exceeded from 4.5 to 2.5×10^{-10} percent of the time. From Table I we see that $\sqrt{2} \leqslant \alpha \leqslant 7$ covers a wide range of practical applications. Using (17), we can fit a parabolic curve to the values of F/σ; using least squares curve fitting, we get

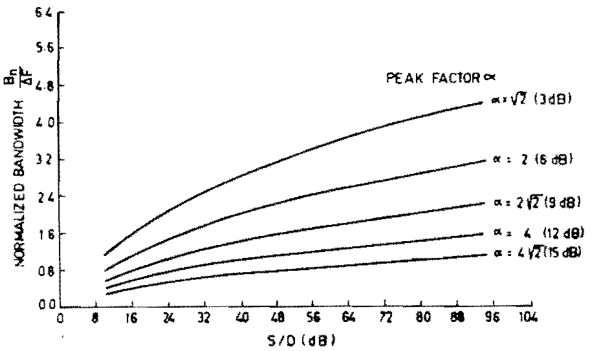

Fig. 1. Normalized bandwidth $B_n/\Delta F$ versus signal-to-band-limiting distortion S/D in dB for several values of the "peak factor" α. Carson's rule bandwidth corresponds to $B_n/\Delta F \simeq 1$.

$$\frac{S}{D} = a(F/\sigma)^2 + b \tag{21}$$

where a and b are 2.3 and 4, respectively. In applying the least squares fitting, ten points were used in the range $\sqrt{2} \leq (F/\sigma) \leq 7$. Thus a wide range of practical situation is covered since $F/\sigma \approx \alpha$, for high-deviation modulation. Finally, (21) can be rearranged as

$$\frac{B_n}{\Delta F} = \frac{0.658}{\alpha} \sqrt{\frac{S}{D} - 4}. \tag{22}$$

Equation (22) is the required relation which gives the necessary bandwidth B_n as a function of the peak deviation ΔF and the peak factor α. A plot of the normalized bandwidth $B_n/\Delta F$ versus S/D is depicted in Fig. 1 for several values of the perak factor α.

Conclusion

A simple relation between the necessary bandwidth and band-limiting distortion has been presented. The proposed relation is intended for Gaussian modulations with large modulation indices. As can be seen from Fig. 1, the Carson's rule bandwidth $B_n \simeq \Delta F$ (for high-index modulation) leads to a serious distortion for small peak factors of the modulating waveform. On the other hand, a waste of spectrum is expected when the peak factor is high. The proposed formula is quite useful for calculating the necessary bandwidths of high-index FM systems. However, it should not be used to calculate adjacent-channel interference since the Gaussian approximation of the spectrum falls off fast at the spectrum tails [10].

References

[1] P. M. Woodward, "The spectrum of random frequency modulation," Telecommunication Research Establishment, Great Malvern, Worcs., England, Memo 666, Dec. 1952.
[2] N. M. Blachman, "Limiting frequency-modulation spectra," *Inform. Contr.*, vol. 1, pp. 26-37, Sept. 1957.
[3] N. M. Blachman and G. A. McAlpine, "The spectrum of a high index GFM waveform: Woodward's theorem revisited," *IEEE Trans. Commun. Technol.*, vol. COM-17, No. 2, pp. 201-208, Apr. 1969.
[4] H. E. Rowe, *Signals and Noise in Communication Systems.* Princeton, NJ: Van Nostrand, 1965, ch. 4.
[5] N. M. Blachman, *Noise and its Effect on Communication.* New York: McGraw-Hill, 1966, p. 46.
[6] P. B. Johns and T. R. Rowbotham, *Communication Systems Analysis.* London: Butterworths, 1972, p. 66.
[7] Bell Telephone Laboratories, *Transmission Systems for Communications*, 1971, pp. 154-55.
[8] N. Abramson, "Bandwidth and spectra of phase and frequency modulated waves," *IEEE Trans. Commun. Syst.*, vol. CS-11, pp. 407-414, Dec. 1963.
[9] S. Haykin, *Communication Systems.* New York: Wiley, 1978, pp. 596-598.
[10] M. Abramowitz and I. A. Stegun, Eds., *Handbook of Mathematical Functions.* Dover, 1970, pp. 968-972.
[11] V. K. Prabhu, "Spectral density bounds on an FM wave," *IEEE Trans. Commun.*, vol. COM-20, no. 5, pp. 980-984, Oct. 1972.

Pyroelectric Materials for Infrared Detectors

SIDNEY B. LANG AND ERNEST H. PUTLEY

We are preparing a historical account of the research and development of pyroelectric materials as sensors for infrared detectors, with special emphasis on the period from 1940 to 1960.

We would greatly appreciate receiving information on this subject.

Manuscript received April 19, 1982.

S. B. Lang is with the Department of Chemical Engineering, Ben Gurion University of the Negev, 84120 Beer Sheva, Israel.

E. H. Putley is with the Royal Signals and Radar Establishment, Great Malvern, Worcs. WR14 3PS, Great Britian.

0018-9219/83/0100-0186$01.00 © 1983 IEEE